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Generalized Monotone Iterative Method for Initial Value Problems

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Abstract—The method of upper, lower solutions, and coupled upper, lower solutions together with the monotone iterative technique yields the monotone sequences or alternating sequences when the forcing function is the sum of two monotone functions.

1. INTRODUCTION

It is well known that the method of upper and lower solutions coupled with the monotone iterative technique offers theoretical as well as constructive existence results in a closed set that is generated by upper and lower solutions. See [1–3] for details. Concerning known results for the first-order initial value problem (IVP for short)

$$u' = F(t,u), \quad u(0) = u_0, \qquad \text{on } J = [0,T],$$

see the "Preview" section of [4]. We observe that the above results cannot be directly applied to the simple logistic equation

$$u' = au - bu^2, \qquad u(t_0) = u_0,$$

where a, b are positive constants.

The natural question is whether it is possible to extend the monotone method when the forcing function is the difference of two monotone functions, so that we obtain some known results as special cases and some new results. The answer is affirmative. This presents a new look into the monotone method results developed so far and also unifies all the results in a single set up. For this, we consider the initial value problem

$$u' = f(t,u) + g(t,u), \quad u(0) = u_0, \qquad \text{on } J = [0,T].$$

This leads to the possibility of having the following four types of upper and lower solutions.

DEFINITION 1.1. *The functions $\alpha, \beta \in C^1[J,R]$ are said to be*

(a) *natural lower and upper solutions if*

$$\begin{aligned} \alpha' &\le f(t,\alpha) + g(t,\alpha), \quad \alpha(0) \le u_0, \qquad \text{on } J, \\ \beta' &\ge f(t,\beta) + g(t,\beta), \quad \beta(0) \ge u_0, \qquad \text{on } J; \end{aligned}$$

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(b) *coupled lower and upper solutions of type I, if*

$$\begin{aligned} &\alpha' \le f(t,\alpha) + g(t,\beta), \quad \alpha(0) \le u_0, \quad && \text{on } J, \\ &\beta' \ge f(t,\beta) + g(t,\alpha), \quad \beta(0) \ge u_0, \quad && \text{on } J; \end{aligned}$$

(c) *coupled lower and upper solutions of type II, if*

$$\begin{aligned} &\alpha' \le f(t,\beta) + g(t,\alpha), \quad \alpha(0) \le u_0, \quad && \text{on } J, \\ &\beta' \ge f(t,\alpha) + g(t,\beta), \quad \beta(0) \ge u_0, \quad && \text{on } J; \end{aligned}$$

(d) *coupled lower and upper solutions of type III, if*

$$\begin{aligned} &\alpha' \le f(t,\beta) + g(t,\beta), \quad \alpha(0) \le u_0, \quad && \text{on } J, \\ &\beta' \ge f(t,\alpha) + g(t,\alpha), \quad \beta(0) \ge u_0, \quad && \text{on } J. \end{aligned}$$

Now corresponding to each type of lower and upper solutions (a), (b), (c), and (d), we can develop four types of sequences. They are

(i)

$$\begin{aligned} &\alpha'_{n+1} = f(t,\alpha_n) + g(t,\alpha_n), \quad \alpha_{n+1}(0) = u_0, \quad && \text{on } J, \\ &\beta'_{n+1} = f(t,\beta_n) + g(t,\beta_n), \quad \beta_{n+1}(0) = u_0, \quad && \text{on } J; \end{aligned}$$

(ii)

$$\begin{aligned} &\alpha'_{n+1} = f(t,\alpha_n) + g(t,\beta_n), \quad \alpha_{n+1}(0) = u_0, \quad && \text{on } J, \\ &\beta'_{n+1} = f(t,\beta_n) + g(t,\alpha_n), \quad \beta_{n+1}(0) = u_0, \quad && \text{on } J; \end{aligned}$$

(iii)

$$\begin{aligned} &\alpha'_{n+1} = f(t,\beta_n) + g(t,\alpha_n), \quad \alpha_{n+1}(0) = u_0, \quad && \text{on } J, \\ &\beta'_{n+1} = f(t,\alpha_n) + g(t,\beta_n), \quad \beta_{n+1}(0) = u_0, \quad && \text{on } J; \end{aligned}$$

(iv)

$$\begin{aligned} &\alpha'_{n+1} = f(t,\beta_n) + g(t,\beta_n), \quad \alpha_{n+1}(0) = u_0, \quad && \text{on } J, \\ &\beta'_{n+1} = f(t,\alpha_n) + g(t,\alpha_n), \quad \beta_{n+1}(0) = u_0, \quad && \text{on } J. \end{aligned}$$

However, it was discovered in [4] that only two of those possible sequences are meaningful. The two sequences considered are

$$\alpha'_{n+1} = f(t,\alpha_n) + g(t,\beta_n), \quad \alpha_{n+1}(0) = u_0, \quad \text{on } J, \tag{1.1}$$
$$\beta'_{n+1} = f(t,\beta_n) + g(t,\alpha_n), \quad \beta_{n+1}(0) = u_0, \quad \text{on } J, \tag{1.2}$$

and

$$\alpha'_{n+1} = f(t,\beta_n) + g(t,\alpha_n), \quad \alpha_{n+1}(0) = u_0, \quad \text{on } J, \tag{1.3}$$
$$\beta'_{n+1} = f(t,\alpha_n) + g(t,\beta_n), \quad \beta_{n+1}(0) = u_0, \quad \text{on } J. \tag{1.4}$$

In Theorem 4 of [4], they have used iterates (1.1) and (1.2) under natural upper and lower solutions with extra assumptions that $\alpha_0(t) \leq \alpha_1(t)$ and $\beta_1(t) \leq \beta_0(t)$ on J and have obtained the same conclusion as in our main result of Theorem 2.1. Similarly, in Theorem 5 of [4], they have used equations (1.3),(1.4) and the natural upper and lower solutions resulting in alternating sequences with extra assumptions $\alpha_0(t) \leq \alpha_2(t)$ and $\beta_2(t) \leq \beta_0(t)$ on J, whereas we have obtained a more general conclusion using coupled upper and lower solutions and iterates (1.3) and (1.4). To be precise, we have obtained intertwined sequences. Note that the alternating sequences obtained are a special case of the intertwined sequences. Similarly, in Theorems 6 and 7 of [4], they needed extra assumptions when coupled upper and lower solutions of type II were utilized. In this paper, we consider coupled lower and upper solutions of type I and II and develop two sequences which converge to coupled minimal and maximal solutions, respectively. In the first case, we get a natural sequence, and in the second case, we get coupled intertwining alternating sequences. However, neither of these results requires an extra assumption as in Theorems 4–6 of [4]. In this framework, all the earlier results can be obtained as special cases and offer many new results.

2. MAIN RESULTS

In this section, we consider the initial value problem

$$u' = f(t,u) + g(t,u), \quad u(0) = u_0, \qquad \text{on } J = [0,T], \quad T > 0, \tag{2.1}$$

where $f, g \in C[J \times R, R]$.

We will develop two theorems for the pair of upper and lower solutions of type I. In Theorem 2.1 below, we will end up with natural monotone sequences starting from α_0, and β_0, which are the coupled lower and upper solutions of type I of equation (2.1). Conclusions of Theorem 2.1 were discussed in [4]. However, we will include it here for completion. In Theorem 2.2, we will start with coupled lower and upper solutions of type I, and we will end up with intertwining alternating sequences with no extra assumptions. Although the hypotheses of Theorems 2.1 and 2.2 are the same, the conclusions are different. The results differ depending on the iterative schemes used to develop the sequences.

THEOREM 2.1. *Assume that*

(i) $\alpha_0, \beta_0 \in C'[J,R]$ *are the coupled lower and upper solutions of type I with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(ii) $f, g \in C[J \times R, R]$, $f(t,u)$ *is nondecreasing in* u *and* $g(t,u)$ *is nonincreasing in* u *on* J.

Then there exists monotone sequences $\alpha_n(t)$ *and* $\beta_n(t)$ *on* J *such that* $\alpha_n(t) \to \rho(t)$ *and* $\beta_n(t) \to r(t)$ *uniformly and monotonically and* (ρ, r) *are coupled minimal and maximal solutions, respectively, to equation (2.1). That is,* (ρ, r) *satisfy*

$$\rho' = f(t,\rho) + g(t,r), \quad \rho(0) = u_0, \qquad \text{on } J, \tag{2.2}$$

$$r' = f(t,r) + g(t,\rho), \quad r(0) = u_0, \qquad \text{on } J, \tag{2.3}$$

where the iterative scheme is given by

$$\alpha_{n+1}' = f(t,\alpha_n) + g(t,\beta_n), \quad \alpha_{n+1}(0) = u_0, \qquad \text{on } J, \tag{2.4}$$

$$\beta_{n+1}' = f(t,\beta_n) + g(t,\alpha_n), \quad \beta_{n+1}(0) = u_0, \qquad \text{on } J. \tag{2.5}$$

PROOF. It is easy to see that the solutions of the linear initial value problems (2.4) and (2.5) exist and are unique for each $k = 1, 2, \ldots$. We will prove that $\alpha_k, \beta_k \in [\alpha_0, \beta_0] = \Omega = [u \in C[J,R] : \alpha_0(t) \leq u \leq \beta_0(t),\ t \in J]$, with $\alpha_k \leq \beta_k$ for each $k \geq 1$. Our aim is to show that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq \beta_k \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0, \qquad \text{on } J. \tag{2.6}$$

We claim that $\alpha_0 \le \alpha_1$ and $\beta_0 \ge \beta_1$. For this purpose, let $p(t) = \alpha_0 - \alpha_1$, then $p'(t) = \alpha_0' - \alpha_1' \le f(t,\alpha_0) + g(t,\beta_0) - f(t,\alpha_0) - g(t,\beta_0) = 0$. This implies that $p'(t) \le 0$ and $p(0) = \alpha_0(0) - \alpha_1(0) \le u_0 - u_0 = 0$. Hence, $p(0) \le 0$. It follows that $p(t) \le 0$ on J. This proves that $\alpha_0(t) \le \alpha_1(t)$ on J. Similarly, we can show $\beta_0 \ge \beta_1$. Furthermore, we claim that $\alpha_1 \le \beta_1$. For that purpose, set $p(t) = \alpha_1 - \beta_1$, then $p'(t) = \alpha_1' - \beta_1' = f(t,\alpha_0) + g(t,\beta_0) - f(t,\beta_0) - g(t,\alpha_0) \le 0$, using the monotone nature of f and g and the fact that $\alpha_0 \le \beta_0$. It follows that $p(t) \le 0$ on J since $p(0) = 0$ on J. This proves that $\alpha_1(t) \le \beta_1(t)$ on J. Thus, we have shown that $\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t)$ holds on J. Hence, (2.6) is true for $k = 1$.

Now assume that (2.6) holds for some $k > 1$ on J. Then all we need to show is that (2.6) holds for $k+1$. Thus, we need to show that $\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k$ holds on J. For this purpose, let $p(t) = \alpha_k - \alpha_{k+1}$ and note that $p(0) = \alpha_k(0) - \alpha_{k+1}(0) = 0$. We get $p'(t) = \alpha_k' - \alpha_{k+1}' = f(t,\alpha_{k-1}) + g(t,\beta_{k-1}) - f(t,\alpha_k) - g(t,\beta_k) \le 0$ using the monotone nature of f and g. This proves $\alpha_k(t) \le \alpha_{k+1}(t)$. Similarly, we can prove that $\beta_{k+1}(t) \le \beta_k(t)$. Now we need to prove that $\alpha_{k+1} \le \beta_{k+1}$ on J. Consider $p(t) = \alpha_{k+1} - \beta_{k+1}$ and note that $p(0) = 0$. Then we get that $p'(t) = \alpha_{k+1}' - \beta_{k+1}' = f(t,\alpha_k) + g(t,\beta_k) - f(t,\beta_k) - g(t,\alpha_k) \le 0$ using the monotone nature of f and g. This proves that (2.6) holds for $k+1$. Hence, (2.6) is valid for all $k = 1, 2, \ldots$.

Also, the sequences $\alpha_k(t), \beta_k(t)$ can be shown to be equicontinuous and uniformly bounded. Thus, by Ascoli-Arzela's theorem, subsequences $\alpha_k(t), \beta_k(t)$ converge to $\rho(t)$ and $r(t)$, respectively, on J. Since the sequences $\alpha_k(t), \beta_k(t)$ are monotone, the entire sequences converge uniformly and monotonically to $\rho(t)$ and $r(t)$, respectively, on J. Therefore, $\rho(t)$ and $r(t)$ satisfy the initial value problems (2.2) and (2.3).

Finally, we claim that ρ and r are coupled minimal and maximal solutions of (2.1). Suppose that u is any solution of 2.1, such that $\alpha_0(t) \le u(t) \le \beta_0(t)$ on J, then we will prove that

$$\alpha_0(t) \le \rho(t) \le u(t) \le r(t) \le \beta_0(t), \qquad \text{on } J.$$

First we will show that

$$\alpha_k \le u \le \beta_k \tag{2.7}$$

holds for any $k \ge 1$ on J. We need to show that (2.7) is true for $k = 1$. For this purpose, let $p(t) = \alpha_1 - u$, we get $p'(t) = \alpha_1' - u' = f(t,\alpha_0) + g(t,\beta_0) - f(t,u) - g(t,u) \le 0$, which implies that $\alpha_1 \le u$. Similarly, we can show that $\beta_1 \ge u$. This proves that (2.7) holds for $k = 1$. We will assume now that (2.7) is true for some $k > 1$. We need to show that it is true for $k+1$. Setting $p(t) = \alpha_{k+1} - u$, we get $p'(t) = \alpha_{k+1}'(t) - u'(t) = f(t,\alpha_k) + g(t,\beta_k) - f(t,u) - g(t,u) \le 0$ on J, using the assumption $\alpha_k \le u \le \beta_k$ holds for some $k > 1$ and the monotone nature of f and g. Since $p(0) = 0$ on J, it follows that $p(t) \le 0$, which proves $\alpha_{k+1}(t) \le u(t)$ on J. Similarly, one can prove that $u(t) \le \beta_{k+1}(t)$ on J. Hence, by induction it follows that $\alpha_k(t) \le u(t) \le \beta_k(t)$ on J for all $k \ge 1$. Now taking the limit as $k \to \infty$, we get $\rho(t) \le u(t) \le r(t)$ on J. This completes the proof.

THEOREM 2.2. *Let (i) and (ii) of Theorem 2.1 hold. Then for any solution $u(t)$ of equation (2.1) with $\alpha_0(t) \le u \le \beta_0(t)$ on J, we get the alternating sequences $\{\alpha_{2n}, \beta_{2n+1}\}$ and $\{\beta_{2n}, \alpha_{2n+1}\}$ satisfying*

$$\alpha_0 \le \beta_1 \le \cdots \le \alpha_{2n} \le \beta_{2n+1} \le u \le \alpha_{2n+1} \le \beta_{2n} \le \cdots \le \alpha_1 \le \beta_0, \tag{2.8}$$

for every $n \ge 1$, where the iterative scheme is given by

$$\begin{aligned} \alpha_{n+1}' &= f(t,\beta_n) + g(t,\alpha_n), & \alpha_{n+1}(0) &= u_0, & &\text{on } J, \\ \beta_{n+1}' &= f(t,\alpha_n) + g(t,\beta_n), & \beta_{n+1}(0) &= u_0, & &\text{on } J. \end{aligned}$$

Moreover, the monotone sequences $\{\alpha_{2n}, \beta_{2n+1}\}$ converge to ρ and $\{\beta_{2n}, \alpha_{2n+1}\}$ converge to r on J, where (ρ, r) are coupled minimal and maximal solutions of equation (2.1), respectively, *satisfying the coupled system*

$$\rho' = f(t,\rho) + g(t,r), \quad \rho(0) = u_0, \qquad \text{on } J, \tag{2.9}$$

$$r' = f(t,r) + g(t,\rho), \quad r(0) = u_0, \qquad \text{on } J. \tag{2.10}$$

Also $\rho \le u \le r$ *on* J.

PROOF. We claim that $\alpha_0 \le \alpha_1$ and $\beta_0 \ge \beta_1$. For this purpose, let $p(t) = \alpha_0(t) - \alpha_1(t)$, then $p'(t) = \alpha_0' - \alpha_1' \le f(t,\alpha_0) + g(t,\beta_0) - f(t,\beta_0) - g(t,\alpha_0) = f(t,\alpha_0) - f(t,\beta_0) + g(t,\beta_0) - g(t,\alpha_0) \le 0$ by the monotone nature of f and g and by the hypothesis $\alpha_0 \le \beta_0$ on J. This implies that $p'(t) \le 0$, and since $p(0) \le 0$, it follows that $p(t) \le 0$ on J. This proves that $\alpha_0(t) \le \alpha_1(t)$ on J. Similarly, we can show $\beta_0 \ge \beta_1$.

Now, our aim is to show that (2.8) is true for $k = 1$. Thus, we need to show that $\alpha_0 \le \beta_1 \le \alpha_2 \le \beta_3 \le u \le \alpha_3 \le \beta_2 \le \alpha_1 \le \beta_0$ holds on J. For this purpose, let u be any solution of (2.1) such that $\alpha_0(t) \le u \le \beta_0(t)$ on J and setting $p(t) = u - \alpha_1$, we get $p'(t) = f(t,u) + g(t,u) - f(\beta_0) - g(\alpha_0) \le 0$ using the monotone nature of f and g and the fact that $\alpha_0 \le u \le \beta_0$. Thus, $p'(t) \le 0$, and since $p(0) = 0$, it follows that $p(t) \le 0$, hence, $u \le \alpha_1$. A similar argument yields $u \ge \beta_1$. In order to avoid repetition, we can prove $u \ge \alpha_2$, $u \le \beta_2$, $u \le \alpha_3$, and $u \ge \beta_3$ in a similar fashion. Now we want to show that $\alpha_0 \le \beta_1 \le \alpha_2 \le \beta_3$, and $\alpha_3 \le \beta_2 \le \alpha_1 \le \beta_0$. Let $p(t) = \alpha_0(t) - \beta_1(t)$, then $p'(t) \le f(t,\alpha_0) + g(t,\beta_0) - f(t,\alpha_0) - g(t,\beta_0) = 0$. This implies that $p'(t) \le 0$ on J, and since $p(0) \le 0$ on J, it follows that $p(t) \le 0$ on J. Thus, $\alpha_0 \le \beta_1$. Similarly, $\alpha_1 \le \beta_0$. Also, let $p(t) = \beta_1 - \alpha_2$, then $p'(t) \le 0$, thus, $p(t) \le 0$, and here we get $\beta_1 \le \alpha_2$. Similarly, we can show $\alpha_1 \ge \beta_2$, $\alpha_2 \le \beta_3$, and $\alpha_3 \le \beta_2$. Hence, we have shown that (2.8) is true for $k = 1$.

Now we will assume that (2.8) holds for some k on J. We use this and we will prove that (2.8) holds for $k+1$ also. Thus, we need to show that the inequality

$$\beta_{2k+1} \le \alpha_{2k+2} \le \beta_{2k+3} \le u \le \alpha_{2k+3} \le \beta_{2k+2} \le \alpha_{2k+1}$$

holds on J. For this purpose, let $p(t) = \beta_{2k+1}(t) - \alpha_{2k+2}(t)$, then we have $p'(t) = f(t,\alpha_{2k}) + g(t,\beta_{2k}) - f(t,\beta_{2k+1}) - g(t,\alpha_{2k+1}) \le 0$ by the monotone nature of f and g and since (2.8) holds for some k. We also have that $p(0) = 0$, thus, $\beta_{2k+1}(t) \le \alpha_{2k+2}(t)$. Similarly, we can show that $\beta_{2k+2}(t) \le \alpha_{2k+1}(t)$, $\alpha_{2k+2}(t) \le \beta_{2k+3}(t)$, and $\alpha_{2k+3}(t) \le \beta_{2n+2}(t)$. We can also show in a similar fashion as before that $u \ge \alpha_{2k+2}$, $u \le \beta_{2k+2}$, $u \ge \beta_{2k+3}$, and $u \le \alpha_{2k+3}$ on J. Hence, by induction, (2.8) is valid for all $k = 0, 1, 2, \dots$.

Also, the sequences $\{\alpha_{2n}, \beta_{2n+1}\}$ and $\{\beta_{2n}, \alpha_{2n+1}\}$ can be shown to be equicontinuous and uniformly bounded. Thus, by the Ascoli-Arzela theorem, subsequences $\{\alpha_{2n_k}, \beta_{2n+1_k}\}$ and $\{\beta_{2n_k}, \alpha_{2n+1_k}\}$ converge to $\rho(t)$ and $r(t)$, respectively, on J. Since the sequences $\{\alpha_{2n}, \beta_{2n+1}\}$ and $\{\beta_{2n}, \alpha_{2n+1}\}$ are monotone, the entire sequences converge uniformly and monotonically to $\rho(t)$ and $r(t)$, respectively, on J. Thus, the coupled system (2.9),(2.10) is satisfied.

Finally, we can show that $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of equation (2.1), such that $\alpha_0(t) \le u(t) \le \beta_0(t)$ on J, then $\alpha_0(t) \le \rho(t) \le u(t) \le r(t) \le \beta_0(t)$. We have already shown that $\alpha_{2k} \le \beta_{2k+1} \le u \le \alpha_{2k+1} \le \beta_{2k}$ holds on J for all $k \ge 1$. Now taking the limit as $k \to \infty$, we get $\rho(t) \le u(t) \le r(t)$ on J. This completes the proof.

COROLLARY 2.1. *In the hypothesis of Theorem 2.2, let* $g(t,u) \equiv 0$. *Then the results of Theorem 2.2 will hold.*

PROOF. To show this, we follow the same procedures as in Theorems 2.1 and 2.2. We will only show a few steps to avoid repetition. We will begin by proving that $\alpha_0 \le \alpha_1$ and $\beta_0 \ge \beta_1$ on J. For this purpose, let $p(t) = \alpha_0 - \alpha_1$. It follows that $p'(t) = \alpha_0' - \alpha_1' \le f(t,\alpha_0) - f(t,\beta_0) \le 0$ by the increasing nature of f and since $\alpha_0 \le \beta_0$. It follows that $p'(t) \le 0$, and since $p(0) \le 0$ on J, we have that $p(t) \le 0$ on J. Thus, $\alpha_0 \le \alpha_1$ on J. Similarly, we can show $\beta_0 \ge \beta_1$ on J. Furthermore, let $p = \beta_1 - \alpha_1$. Then $p'(t) = f(t,\alpha_0) - f(t,\beta_0) \le 0$ by the increasing nature of f.

Thus, $p'(t) \leq 0$, and since $p(0) = 0$ on J, we get $p(t) \leq 0$ on J. Hence, $\beta_1 \leq \alpha_1$ on J. Continuing in this fashion, we can prove that (2.8) holds.

REMARK 2.1. In [2], the monotone method was developed when the forcing function was either increasing or could be made increasing by adding a linear function. In this case, we had natural sequences which converged to the extremal solutions. However, if the forcing function was decreasing, then we had alternating sequences under an additional assumption. In [4], an attempt was made to exhaust all possible coupled upper or lower solutions. However, results of Theorem 2.2 were not included. From our results of Theorems 2.1 and 2.2, we can conclude that there exists a natural sequence or an intertwining alternating sequence without any extra assumption depending on the formula that is used to develop the sequences. Further, we have also proved that the natural sequences and the alternating sequences do not depend on whether or not the forcing function is increasing or deceasing.

REMARKS 2.2.

(i) If $g(t, u) \equiv 0$, then we will get the results from Theorems 2.1 and 2.2 for f increasing and $M \equiv 0$.

(ii) If $g(t, u) \equiv 0$, and f is not increasing in u, then $F(t, u) = f(t, u) + Mu$ is nondecreasing on u for some $M > 0$. Then one can consider the IVP

$$u' + Mu = F(t, u), \quad u(0) = u_0, \qquad \text{on } J,$$

where we get

$$\begin{aligned} e^{mt}(u' + Mu) &= e^{mt}F(t, u), \\ (e^{mt}u)' &= e^{mt}F(t, u). \end{aligned}$$

Using the transformations $\tilde{u} = ue^{mt}$ and $\tilde{F}(t, u) = e^{mt}f(t, u)$, we get

$$\tilde{u}' = \tilde{F}\left(t, \tilde{u}e^{-mt}\right).$$

Furthermore, we have

$$\begin{aligned} v_0' &\leq f(t, v_0), \\ v_0' + Mv_0 &\leq f(t, v_0) + Mv_0, \\ \left(e^{mt}v\right)' &\leq e^{mt}F(t, v_0), \\ \tilde{v_0}' &\leq e^{mt}F(t, v_0), \\ \tilde{v_0}' &\leq \tilde{F}\left(t, \tilde{v_0}, e^{-mt}\right). \end{aligned}$$

Also, $\tilde{w_0}' \geq \tilde{F}(t, \tilde{w_0}e^{-mt})$. So clearly, we have that $\tilde{v_0}$ is a lower solution and $\tilde{w_0}$ is an upper solution, thus, we have the same conclusion as in Remark 2.2(i).

REMARK 2.3. We can always construct coupled upper and lower solutions of type II as in this paper when $f(t, u)$ and $g(t, u)$ is nondecreasing and nonincreasing, respectively. We state this result below as a lemma. In this case where we can construct upper and lower solutions, we can develop results corresponding to Theorems 2.1 and 2.2 of this paper with an additional assumption. We will merely state the results without proof. See [4] for details of all the results presented below.

LEMMA 2.1. *Suppose that $f(t, u)$, $g(t, u)$ are monotone nondecreasing and monotone nonincreasing in u, respectively, on J, then there exists coupled lower and upper solutions of type II for (2.1).*

To avoid monotony, we just indicate the proofs of the next two theorems since it follows on the same lines as Theorems 2.1 and 2.2, respectively, and the earlier known results [2,4]. Hence, we state the theorems without proof.

THEOREM 2.3. *Assume the hypothesis of Lemma 2.1 holds, and let α_0 and β_0 be coupled lower and upper solutions, respectively, of type II with $\alpha_0(t) \le \beta_0(t)$ on J. Further, starting from α_0 and β_0, if the iterative scheme is constructed by*

$$\alpha'_{n+1} = f(t,\alpha_n) + g(t,\beta_n), \quad \alpha_{n+1}(0) = u_0, \qquad \text{on } J, \tag{2.11}$$

$$\beta'_{n+1} = f(t,\beta_n) + g(t,\alpha_n), \quad \beta_{n+1}(0) = u_0, \qquad \text{on } J, \tag{2.12}$$

then the conclusions of Theorem 2.1 hold provided $\alpha_0 \le \alpha_1$ and $\beta_0 \ge \beta_1$ on J.

PROOF. From hypothesis we have $\alpha_0 \le \alpha_1$ and $\beta_0 \ge \beta_1$ on J. We can prove $\alpha_1 \le \beta_1$ just as in the proof of Theorem 2.1. Now one can imitate the proof of Theorem 2.1.

THEOREM 2.4. *Assume the hypothesis of Lemma 2.1 holds, and let α_0 and β_0 be coupled lower and upper solutions, respectively, of type II with $\alpha_0(t) \le \beta_0(t)$ on J. Further, starting from α_0 and β_0, if the iterative schemes are developed by*

$$\alpha'_{n+1} = f(t,\beta_n) + g(t,\alpha_n), \quad \alpha_{n+1}(0) = u_0, \qquad \text{on } J,$$

$$\beta'_{n+1} = f(t,\alpha_n) + g(t,\beta_n), \quad \beta_{n+1}(0) = u_0, \qquad \text{on } J,$$

then the conclusions of Theorem 2.2 hold provided $\alpha_0 \le \beta_1$ and $\beta_0 \ge \alpha_1$.

PROOF. One can imitate the proof of Theorem 2.2 by using the hypothesis $\alpha_0 \le \beta_1$ and $\beta_0 \ge \alpha_1$.

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